The New ISO/WD 16355 Standard, Transfer Functions and the Effect of Ratio Scale in QFD

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Abstract: *The new ISO/WD 16355 Working Draft defines Quality Function Deployment (QFD) as a statistical tool for analyzing customer's voice and deploy it into technical solutions. This has a huge impact on the traditional way of using QFD. Instead of symbols for the correlation matrix, its cells contain ratio scale numbers. Instead of working with relative weights, profile vectors allow combining and comparing customer's needs and technical solutions.*

Transfer functions are a concept from signal theory. Transfer functions derive from some measured response, e.g., the voice of the customer goal profile, what solutions can control such a response. The technical solution usually is unknown and needs optimization. Moreover, it never meets the goal profile exactly. The difference between expected and achieved response is measurable as convergence gap. The crucial requirement for such derived measurements is to know the statistical variation that is tolerable.

Based on the new ISO/WD 16355 working draft, goal profiles compare and combine as vectors. This opens the way for the vision that Prof. Akao created in the 1980ies, namely that comprehensive QFD transfer functions constitute a network of interrelated processes implementing a value chain.

Keywords: Quality Function Deployment; Six Sigma; Multilinear Transfer Functions; Perron-Frobenius Theorem

1 Introduction

The impact of combining QFDs in the new worlds of the Internet of things is substantial.

Moreover, the QFD matrix becomes a measuring tool for big data. It is simple; you have to learn how to ask the right question. These questions constitute your experimental solution controls. Fill data into the cells that traditionally hold three symbols or nothing, count it and now find out whether the observed response matches in some way the measured response. If so, you likely have asked the right questions and you can rely upon the solution profile as answers.

Prof. Akao's vision (Akao, 1990) has become reality and accessible to all, thanks to the work of the technical committee that drafted the ISO/WD 16355 working draft. This paper focuses on some of the new ways to use QFD as a statistical tool, what the future of QFD is likely to be. QFD used as a mathematical discipline makes it possible to create machine-learning applications that use QFD in real time to adjust services, and even the product, to the rapidly changing customer's needs, or the market. In addition, quality management becomes real-time, when combined with new cloud technologies.

This paper relies on a new book to appear from the authors that explains the mysteries of *Six Sigma Transfer Functions* used in this paper (Fehlmann & Kranich, 2015 (to appear)).

2 Modern QFD

The main changes to traditional QFD that are apparent are the dismissal of symbols in cells and their replacement by numbers. Although it is still possible to use a restricted set in QFD workshops, for the sake of enforcing decisions, for instance the traditional 1-3-9 scale, Saaty's postulate for a *Ratio Scale* is now compulsory. Thus, for instance $9 = 3 * 3$ holds; moreover, intermediate values are admissible. If symbols remain in use, for instance in QFD workshops, the working draft proposes standard five or nine levels. The roof also disappeared, and the various possible QFD matrices have become functions that transfer prioritization and quantification from one information set into another.

This is a reference to transfer functions. This are probably the most powerful tools that mathematics developed in the 20th century. Transfer functions are used everywhere: from converting analogue to digital music forth and back, analyze signals, predict customer behavior, to detecting exoplanets. Outstanding application examples of this theory are the calculation of Google's *PageRank* that propelled ICT from a scientific niche into mainstream economy, see Gallardo (Gallardo, 2007), and Langville and Meyer (Langville & Meyer., 2006), and Saaty's *Analytic Hierarchy Process* (AHP), see Saaty (Saaty, 1990), (Saaty & Peniwati, 2008).

2.1 Modern QFD as Transfer Functions

Transfer function between the real vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are linear mappings between vector spaces of the form $f: \mathbb{R}^n \to \mathbb{R}^m$ with $\boldsymbol{y} = f(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^m$. QFD names $\boldsymbol{x}$ the *Cause*, $\boldsymbol{y}$ the *Effect*. Usually, the effect is observable and even measurable – by going to the Gemba (Mazur & Bylund, 2009), or listening to customer's voice by the Net Promoter method (Fehlmann & Kranich, 2014) – thus meeting customer's needs, or match business drivers that make a product successful, while the cause remains unknown. The QFD method uncovers possible causes. The challenge is to solve the equation

$$\boldsymbol{y} = f(\boldsymbol{x}) \tag{1}$$

for the unknown $\boldsymbol{x} \in \mathbb{R}^n$ and a transfer function f that is detectable using Six Sigma measurements, e.g., by the *Design of Experiments* discipline, or based on the QFD teams' collective expertise.

2.2 Linear Algebra Basics

In order to understand the impact the proposed new ISO/WD 16355 standard, it is useful to have a look at linear algebra; see Fehlmann (Fehlmann, 2003) and (Fehlmann, 2005). The solution vector $\boldsymbol{x}$ is termed the solution profile and $\boldsymbol{y}$ the response profile. The challenge is, to find a solution profile $\boldsymbol{x}$ that produces a response profile $\boldsymbol{y} = f(\boldsymbol{x})$ that is close enough to some *Target* or *Goal Profile* τ_y; i.e., minimize the distance between τ_y and $\boldsymbol{y}$.

Since all consist the real-valued column vector:

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_m \end{pmatrix} = \langle y_1, y_2, \dots, y_m \rangle \tag{2}$$

Conformant with the literature, and in order to avoid writing columns vectors vertically throughout this paper, the writing convention $y = \langle y_1, y_2, \dots, y_m \rangle$ applies for representing column vectors.

In order to solve equation (1), let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear functional. There exists a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ of dimension $m \times n$ with $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$; see textbooks on linear algebra, e.g., Meyer (Meyer, 2000).

Hence,

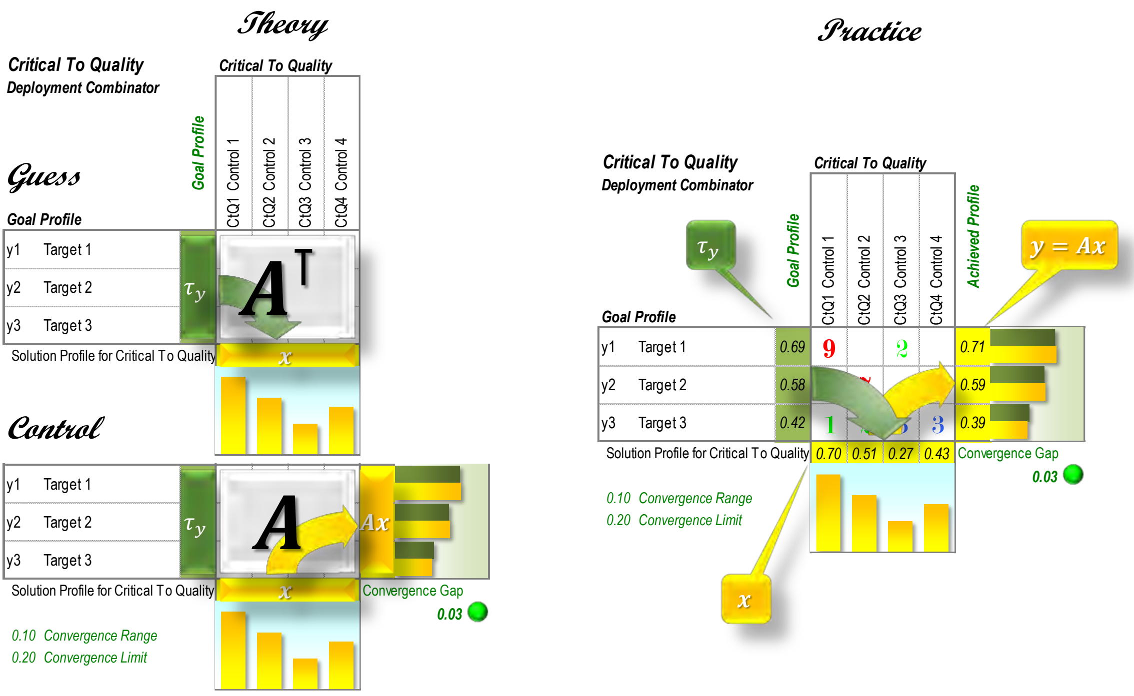
$$\boldsymbol{y} = f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \cdot (x_1, x_2, \dots, x_n) = \begin{pmatrix} y_1 = \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ y_m = \sum_{j=1}^{n} a_{mj} x_j \end{pmatrix} \tag{3}$$

reveals how to calculate the i^{th} component of $\boldsymbol{A}\boldsymbol{x}$. The matrix $\boldsymbol{A}$ multiplies with the solution profile vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ by rows. Thus, the response profile $\boldsymbol{y} = \langle y_1, y_2, \dots, y_m \rangle$ is exactly what QFD practitioners do when evaluating a QFD matrix. It tells what response $\boldsymbol{y}$ to expect given a technical solution $\boldsymbol{x}$.

3 Solving a QFD

In today's QFD practice, finding the technical solution controls $\boldsymbol{x}$ still relies on try and error. In the past, there were even QFD that did not check for error but simply believed that their method to evaluate a QFD matrix is correct. Assume some goal profile $\boldsymbol{\tau_y} = \langle \tau_{y_1}, \tau_{y_1}, \dots, \tau_{y_m} \rangle$ is known. The challenge is, finding a solution for the unknown $\boldsymbol{x}$ in the equation $\boldsymbol{\tau_y} = \boldsymbol{A}\boldsymbol{x}$.

Figure 1: Theory and Practice in Traditional QFD Evaluation



Calculating the achieved solution uses equation (3). The achieved solution compares with the original target, the goal profile. The first step is the *Guess* step for finding a solution profile $\boldsymbol{x}$.

The usual method relies on the transpose of matrix $\boldsymbol{A}$**,** usually written as $\boldsymbol{A}^{\top}$, for calculating an approximation of $\boldsymbol{x}$**:**

$$\boldsymbol{x} = \boldsymbol{A}^{\top}\boldsymbol{y} = (x_1, x_2, \dots, x_n)\,, \quad \text{where} \quad x_j = \sum_{i=1}^{m} a_{ij} y_j \quad \text{for } j = 1,2, \dots, n \tag{4}$$

This is the definition of the transpose; see e.g., Meyer's textbook on linear algebra and matrices (Meyer, 2000). Vector multiplication is by columns, as opposed to (3). Figure 1 demonstrates how an Excel tool for calculating a QFD matrix unites both steps to calculate the difference between $\boldsymbol{y}$ and $\boldsymbol{\tau_y}$.

3.1 The Eigenvector Solution Method

Obviously, $\boldsymbol{x} = \boldsymbol{A}^{\top}\boldsymbol{\tau_y}$ is a good solution if $\boldsymbol{y} = \boldsymbol{\tau_y}$. However, this is normally not the case. Why should $\boldsymbol{x} = \boldsymbol{A}^{\top}\boldsymbol{\tau_y}$ yield a solution for $\boldsymbol{\tau_y} = \boldsymbol{A}\boldsymbol{x}$? In general, it does not.

However, in many cases of interest to QFD practitioners, it does. At this point, the theory of *Eigenvectors* and *Eigenvalues* comes into play; see e.g., Kressner (Kressner, 2005). In order to explain eigenvectors and eigenvalues, let $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ be an arbitrary square matrix. From linear algebra it is well known that almost all vectors $\boldsymbol{x} \in \mathbb{R}^{n}$change their directions when the vectors are multiplied by the matrix $\boldsymbol{S}$, see e.g., Lang (Lang, 1973), and Roman (Roman, 2007). A non-zero vector $\boldsymbol{x}$ is called an *Eigenvector* of the matrix $\boldsymbol{S}$, if $\boldsymbol{x}$ and the vector $\boldsymbol{Sx}$ are pointing in the same direction, i.e., eigenvectors are the directions which are invariant under the transformation $\boldsymbol{S}$.

The *Eigenvalue* λ reveals whether the vector $\boldsymbol{Sx}$ remains unchanged (i.e., $\lambda = 1$), is changed in direction (i.e., $\lambda < 0$), is shrunk (i.e., $0 < |\lambda| < 1$), or stretched (i.e., $|\lambda| > 1$). Thus, the fundamental equation to solve an eigenvector respectively eigenvalue problem is

$$\boldsymbol{Sx} = \lambda\boldsymbol{x} \tag{5}$$

A natural question arises: How can the equation above (5) help to solve $\boldsymbol{y} = \boldsymbol{Ax}$, in order to calculate a solution profile $\boldsymbol{x} \in \mathbb{R}^{n}$ with respect to a linear multiple response transfer function $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and a response profile $\boldsymbol{y} \in \mathbb{R}^{m}$?

Following Fehlmann and Kranich (Fehlmann & Kranich, 2015 (to appear)), (Fehlmann & Kranich, 2011), a response profile $\boldsymbol{y}$ can be determined by solving the following eigenvector resp. eigenvalue problem:

$$\boldsymbol{AA}^{\top}\boldsymbol{y} = \lambda\boldsymbol{y} \tag{6}$$

Obviously, the matrix $\boldsymbol{AA}^{\top} \in \mathbb{R}^{m\times m}$ is symmetric, i.e. $\boldsymbol{AA}^{\top} = (\boldsymbol{AA}^{\top})^{\top}$. It is well known from the theory of eigenvalues that this matrix has exactly m (not necessarily) distinct real eigenvalues. There exists a set of m real eigenvectors, one for each eigenvalue, which are mutually orthogonal and thus linear independent, even in the case when the eigenvalues are not distinct. In most cases, the matrix $\boldsymbol{AA}^{\top}$ in (6) is positive definite, i.e., $\boldsymbol{x}^{\top}\boldsymbol{AA}^{\top}\boldsymbol{x} > \boldsymbol{0}$ for all $\boldsymbol{x} \in \mathbb{R}^{n} \setminus \{\boldsymbol{0}\}$, and therefore has a *Principal Eigenvector* $\boldsymbol{y_E}$. The theorem that establishes this is the *Perron-Frobenius Theorem*. We refer to the literature for its exact formulation and proof, e.g., Meyer (Meyer, 2000).

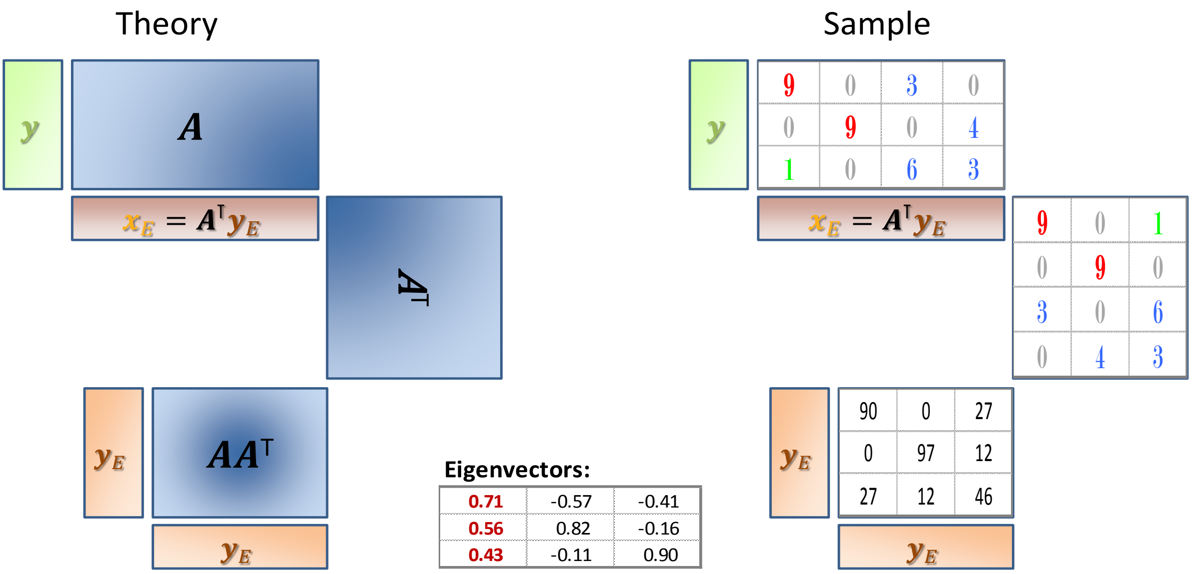
3.2 Calculating the Eigenvector

There are many methods available for calculating eigenvectors; the most popular and best suitable for our complex problem solving techniques is the *Jacobi Iterative Method*. Most mathematical packages contain eigenvector calculation methods; for Microsoft Excel, a free open source tool is available (Volpi & Team, 2007). Consult the book and web site of Robert de Levie (Levie, 2012). Most statistical packages contain the eigenvector methods, e.g., the R project (The R Foundation, 2015).

The solution approach $\boldsymbol{x} = \boldsymbol{A}^{\top}\boldsymbol{\tau_y}$, shown in Figure 2**,** is what QFD practitioners do all the time without knowing whether this yields any valuable result or not. Indeed, those that omit calculating the convergence gap rely on guesses and belief; this is not serious QFD but rather charlatanries. However, experience shows that this approach nevertheless yields valuable solutions. The reason for this is that QFD matrices looking balanced most often have a principal eigenvector, and $\boldsymbol{x} = \boldsymbol{A}^{\top}\boldsymbol{\tau_y}$ is just the first iteration for the Jacobi Iterative Method.

The idea is to revert *cause* and *effect*. First, transpose the matrix and calculate the combined symmetrical square matrix $\boldsymbol{A}\boldsymbol{A}^{\top}$. According the Theorem of Perron-Frobenius, this symmetrical square matrix has a principal eigenvector $\boldsymbol{y_E}$ with $\boldsymbol{y_E} = \boldsymbol{A}\boldsymbol{A}^{\top}\boldsymbol{y_E}$. Using that eigenvector, the solution for $\boldsymbol{A}\boldsymbol{x_E} = \boldsymbol{y_E}$ is $\boldsymbol{x_E} = \boldsymbol{A}^{\top}\boldsymbol{y_E}$. $\|\boldsymbol{y} - \boldsymbol{y_E}\|$ is the *Convergence Gap*; $\boldsymbol{x_E} = \boldsymbol{A}^{\top}\boldsymbol{y_E}$ is called the *Eigencontrols* of $\boldsymbol{A}$. If $\boldsymbol{y}$ happens to be near an eigenvector of $\boldsymbol{A}$, we have an approximate solution for $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\tau_y}$. However, solutions do not exist for all transfer functions $\boldsymbol{A}$.

Figure 2: How the Eigenvector is calculated (Perron-Frobenius Theorem)



As a corollary, one notes that the constraint to positive matrices $\boldsymbol{A}$ can be lowered. Indeed, QFD matrices represented by $\boldsymbol{A}$ can contain negative cells as long as $\boldsymbol{A}\boldsymbol{A}^{\top}$ remains positive definite. For only a few negative values, this is typically the case. As examples, consult for instance the Net Promoter analysis method proposed by the authors in (Fehlmann & Kranich, 2014).

3.3 A Measure for the Closeness of $\boldsymbol{y}$ to $\boldsymbol{\tau_y}$

One crucial question needs clarification. How to measure the distance between the current response profile $\boldsymbol{y}$ and the predefined target profile $\boldsymbol{\tau_y}$? Since we are talking about vector spaces, the Euclidian norm is the metrics of choice. Thus,

$$\|\boldsymbol{y} - \boldsymbol{\tau_y}\| = \sqrt{\sum_{i=1}^{m} \left(y_i - \tau_{y_i}\right)^2} \tag{7}$$

with $\left(\boldsymbol{y} - \boldsymbol{\tau_y}\right)_i = y_i - \tau_{y_i}$, $1 \leq i \leq m$, is termed the *Convergence Gap* and reveals the quality of the approximation of $\boldsymbol{y}$ to $\boldsymbol{\tau_y}$. If this gap fulfills a predefined convergence criterion, then $\boldsymbol{y}$ is (sufficiently) close to $\boldsymbol{\tau_y}$. Then, $\boldsymbol{x} = \boldsymbol{A}^{\top}\,\boldsymbol{\tau_y}$ calculates the approximate solution profile $\boldsymbol{x}$, solving $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$.

3.4 Normalization of Vectors

In general, any vector $\boldsymbol{y}$ can be normalized using the Euclidean Norm (8) bringing them to equal length.

$$\|\boldsymbol{y}\| = \sqrt{\sum_{i=1}^{m} {y_i}^2} \tag{8}$$

When vectors differ not in length but in direction only, it is possible to add, subtract or compare two vectors varies with regard to their direction only. Obviously, this allows adding, subtracting or comparing profiles, and this is yet another practice important for QFD practitioners.

Given a vector $\boldsymbol{y} = \langle y_1, \dots, y_m \rangle$ of dimension m, its normalized variant is

$$\boldsymbol{y}' = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} = \langle \frac{y_1}{\|\boldsymbol{y}\|}, \dots, \frac{y_m}{\|\boldsymbol{y}\|} \rangle \tag{9}$$

However, traditional QFD uses division by its maximum component instead, and that yet another reason, why Mazur declares certain traditional practices as *bad mathematics* (Mazur, 2014). This needs some further consideration. It is worth calling *Statistical Thinking*.

4 Profiles and Weights

Statistical thinking means looking at events as vectors in a multidimensional vector space, avoiding maximizing some favored topic. The difference in normalization can become substantial.

Modern QFD distinguishes *Weights* and *Profiles*.

A *Weight* is a percentage of its total importance, e.g., topic 1 has 5%, topic 2 has 85%, and topic 3 has 10% relative weight in percentage of the total importance (100%), see weight vector (10).

A *Profile* in contrary is a vector of length 1. Their vector components are not relative weights; however, its length always are fixed and thus comparable.

Figure 3 demonstrates how two priority profiles corresponding to the weight vectors (10) and (11) sum up into a combined priority profile, respective weight vector (12). This happens in the three column. In the rows, weight vectors transform into profile vectors and back again.

Figure 3: Adding Weights versus Adding Profiles

	Weights	→	Profiles	→	Weights	
Topic 1	5%	0.00	0.06	0.06	5%	
Topic 2	85%	0.72	0.99	0.99	85%	
Topic 3	10%	0.01	0.12	0.12	10%	
	100%	0.86	1.00	1.17	100%	Weight & Profile 1
	plus↓	→	plus↓	→	Weights	
Topic 1	33%	0.11	0.57	0.57	33%	
Topic 2	34%	0.12	0.59	0.59	34%	
Topic 3	33%	0.11	0.57	0.57	33%	
	100%	0.58	1.00	1.73	100%	Weight & Profile 2
	sum↓		sum↓	→	Weights	
Topic 1	0.38		0.63	0.34	21.7%	
Topic 2	1.19		1.58	0.86	54.5%	
Topic 3	0.43		0.69	0.37	23.7%	
	2.00		1.84	1.58	100%	Sum of Profiles 1+2
	norm↓	→	Profiles	→	Weights	≠
Topic 1	19%	0.04	0.22	0.22	19.0%	
Topic 2	60%	0.35	0.69	0.69	59.5%	
Topic 3	22%	0.05	0.25	0.25	21.5%	
	100%	0.66	0.77	1.17	100%	Sum of Weights 1+2
				0.24		Convergence Gap

Summing up the corresponding weight vectors is bad mathematics; the results are not the same as when summing up the priority profiles and converting them back to weight vectors. In detail, when represented as a vector, the weights looks as follows:

$$\langle 5\%, 85\%, 10\% \rangle \tag{10}$$

The Euclidian length of vector (10) is $\sqrt{0.1^2+0.7^2+0.2^2}=0.86$. Since importance might correlate to the budgeted amount that is being spend for these topics, weight percentage matters.

Take as an example another weight vector, say the second part in Figure 3:

$$\langle 33\%, 34\%, 33\% \rangle \tag{11}$$

Clearly, its Euclidian length is 0.58. Thus, the two weight vectors (10) and (11) have unequal length. Adding them by components – percent becomes decimal fraction – yields $\langle 0.38, 1.19, 0.43 \rangle$.

This corresponds to the weight vector

$$\langle 19.0\%, 59.5\%, 21.5\% \rangle \tag{12}$$

However, since the first weight vector was longer, it contributes more to the sum vector (12) than the second does. Unless there is some special business reason for it, such an unequal treatment can have disastrous results, for instance spending investment money in the wrong places.

Converting both vectors to priority profiles, i.e., by dividing their components through its length, yields two priority vectors of equal length but pointing into two different directions in the vector space. Combining them by adding the three components yields the profile $\langle 0.34, 0.86, 0.37 \rangle$ with Euclidian unit length $\|\langle 0.34, 0.86, 0.37 \rangle\| = 1$. This profile again converts into weights by normalizing the sum of components to 100%.

The result is the weight vector:

$$\langle 21.7\%, 54.5\%, 23.7\% \rangle \tag{13}$$

This obviously is not the same weight vector as (12). The convergence gap is 0.24. For a vector of unit length, this is one quarter of its length, a substantial difference, pointing into another direction. The reason for the difference is the length inequality. In practice, this difference is difficult to detect and might escape the attention of QFD practitioners. The selected example uses extremely different weight vectors to demonstrate the case.

As a conclusion, while it is safe to compare vector profiles, adding, subtracting, or multiplying them as needed in a QFD, it is bad mathematics if you compare, add or subtract weight vectors.

5 Summary

Quality Function Deployment is changing from a workshop practice based on empirical experience to become a mainstream practice based on sound mathematical foundations. Together with other outstanding applications of transfer functions, it marks the fundamental change occurring in economy and quality management of the 21st century. The upcoming ISO/WD 16355 Standard is paving the way for it, even if the exact content is not yet fully fixed and available.

However, despite sound mathematics, the key elements of QFD will remain the collections of customer's needs and business drivers in the market, and the collaboration of expert teams, even if part of it can be automated and might become available in real time. The number and the quality of tools offered to these expert teams is what makes the difference, and this entails cutting away some of the old hats today still in use, and embrace modern mathematics with modern QFD.

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